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Mass in Quantum Yang-Mills Theory (comment on a Clay millenium problem)

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— Dedicated to IMPA on the occasion of its 50^{th} *anniversary*

Abstract. The Clay Millenium Problem on the mass gap for the Quantum Yang-Mills Field Theory is commented upon. Particular emphasis is put on the importance of the dimensional transmutation after the quantisation.

Keywords: Yang-Mills field, mass, dimensional transmutation.

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Among seven problems, proposed for XXI century by Clay Mathematical Institute [1], there are two stemming from physics. One of them is called "Yang-Mills Existence and Mass Gap". The detailed statement of the problem, written by A. Jaffe and E. Witten [2], gives both motivation and exposition of related mathematical results, known until now. Having some experience in the matter, I decided to completement their text by my own personal comments. These comments in no way show the direction for a solution of the problem. However they could be useful for a person who has no background in physical literature but decided to attack the problem.

1 What is Yang-Mills field

Yang-Mills field bears the name of the authors of the famous paper [3], in which it was introduced into physics. From mathematical point of view it is a connection

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in fiber bundle with a compact group G as a structure group. We shall treat the case when the corresponding principal bundle E is trivial

$$E = M_4 \times G$$

and the base M_4 is a four dimensional Minkowski space.

In our setting it is convenient to describe the Yang-Mills field as one-form A on M_4 with the values in the Lie algebra $\mathcal{G}$ of G:

$$A(x) = A^a_\mu(x) t^a dx^\mu .$$

Here x^μ, $\mu = 0, 1, 2, 3$ are coordinates on M_4; t^a, $a = 1, \dots, \dim G$ — basis of generators of $\mathcal{G}$ and we use the traditional convention of taking sum over indices entering twice.

Local rotation of the frame

$$t^a \to g(x) t^a g^{-1}(x),$$

where $g(x)$ is a function on M_4 with the values in G induces the transformation of the A (gauge transformation)

$$A(x) \to g^{-1}(x) A(x) g(x) + g^{-1} dg(x).$$

Important equivalence principle states, that a physical configuration is not a given field A, but rather a class of gauge equivalent fields. This principle essentially uniquely defines the dynamics of the Yang-Mills field.

Indeed, the action functional, leading to the equation of motion via variational principle, must be gauge invariant. Only one local functional of second order in derivatives of A can be constructed.

For that we introduce the curvature-two form with values in $\mathcal{G}$

$$F = dA + A^2,$$

where the second term in RHS is exterior product of one-form and commutator in $\mathcal{G}$. In more detail

$$F = F^a_{\mu\nu} t^a dx^\mu \wedge dx^\nu ,$$

where

$$F^a_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + f^{abc} A^b_\mu A^c_\nu$$

and f^{abc} are structure constants of $\mathcal{G}$ entering the basic commutation relation

$$[t^a, t^b] = f^{abc} t^c.$$

The gauge transformation of F is homogenous

$$F \to g^{-1} F g,$$

so that the 4-form

$$S = \mathrm{tr}\, F \wedge F^* = F^a_{\mu\nu} F^a_{\mu\nu} \omega$$

is gauge invariant. Here F^* is a Hodge dual to F with respect to Minkowskian metric, and ω is corresponding volume element. It is clear, that S contains the derivatives of A at most in second order. The integral

$$\mathcal{A} = \frac{1}{4g^2} \int_{M_4} S$$

can be taken as an action functional. The constant g^2 in front of the integral is a dimensionless parameter which is called a coupling constant. Let us stress, that it is dimensionless only in the case of four dimensional space-time.

Remind that in general the dimension of physical quantity is a product of powers of 3 fundamental dimensions [L] — lenght, [T] — time and [M] — mass with usual units of cm, sec and gr. However in relativistic quantum physics we have two fundamental constants — velocity of light c and Plank constant $\hbar$ and use the convention, that $c = 1$ and $\hbar = 1$, reducing the possible dimensions to the powers of lenght [L]. The Yang-Mills field has dimension $[A] = [\mathrm{L}]^{-1}$, the curvature $[F] = [\mathrm{L}]^{-2}$, the volume element $[\omega] = [\mathrm{L}]^4$, so that $\mathcal{A}$ is dimensionless as it must be, because it has the same dimension as $\hbar$. We see, that $\mathcal{A}$ contains terms in powers of A of degrees 2,3,4

$$\mathcal{A} = \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4$$

which means that Yang-Mills field is selfinteracting.

Among many approaches to quantizing the Yang-Mills theory the most natural is that of the functional integral. Indeed, the equivalence principle is taken into account in this approach by integrating over classes of equivalent fields. There is no place here to explain this purely heuristic method of quantization, moreover it hardly will lead to a solution of Clay Problem. So we shall just write the main formula with hope to appeal to the intuition of the reader. This formula gives a generating functional $Z(A_{\text{as}})$ for physical entities such as S-matrix. The field A_{as} describes the asymptotical behavoir of the fields A, over which we integrate, in time-like directions. Here follows the formula

$$Z(A_{\text{as}}) = \int \exp\{i \int \left(\frac{1}{4g^2} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{1}{2g^2} (\partial_\mu A^a_\mu)^2 + \partial_\mu \bar{c}(x) \nabla_\mu c(x) \right) d^4x\} \prod_{\mu,a,x} dA^a_\mu d\bar{c}^a dc^a. \quad (1)$$

Here the term $\exp \frac{i}{2g^2} \int (\partial_\mu A^a_\mu)^2 d^4x$ takes care of integration along the orbits of the gauge transformation and the last term assures the true normalization of this integration. There enter the variables $\bar{c}^a(x)$, $c^a(x)$ which are the generators of the Grassmann algebra so that the integral over them in Berezin sence [4] gives representation of determinant of operator

$$M = \partial_\mu \nabla_\mu$$

Here ∇_μ is a covariant derivative, acting on c^a as follows

$$\nabla_\mu c^a = \partial_\mu c^a + f^{abc} A^b c^c.$$

The explanation of this formula, first introduced by V. Popov and me [5] can be found in any modern textbook on Quantum Field Theory. I can recommend the text which I coauthored with A. Slavnov [6] or monograph by Peskin and Shroeder [7]. As I already said for the goal of this comments just intuitive grasping of this formula is enough.

2 What is the mass

It was the advent of the special relativity which has given a natural definition of mass. A free massive particle has the following expression of the energy ω in terms of its momentum

$$\omega(p) = \sqrt{p^2 + m^2}.$$

In quantum version mass appears as a parameter (one out of two) of the irreducible representation of the Poincare group (group of motion of the Minkowski space).

In quantum field theory this representation (insofar as m) defines a one-particle space of states $\mathcal{H}_m$ for a particular particle entering the full spectrum of particles. The state vectors in such a space can be described as functions $\psi(p)$ of momentum p and $\omega(p)$ defines the energy operator.

The full space of states has the structure

$$\mathcal{H} = \mathbb{C} \oplus \left(\sum_i \oplus \mathcal{H}_{m_i} \right) \oplus \cdots ,$$

where one dimensional space $\mathbb{C}$ corresponds to the vacuum state and $\cdots$ mean spaces of many-particles states, being tensor products of one-partical spaces. In particular if all particles in the system are massive the energy has zero eigenvalue corresponding to vacuum and then positive continuous spectrum from $\min m_k$ till infinity. In other words the least mass defines the gap in the spectrum. The Clay problem requires the proof of such a gap for the Yang-Mills theory.

We see an immediate difficulty. In the formulation of the classical Yang-Mills theory no dimesional parameter entered. On the other hand, the Clay Problem requires, that in quantum version such parameter must appear. How come?

I decided to write these comments exactly for the explanation how quantization can lead to appearence of dimensional parameter when classical theory does not have it. This possibility is connected with the fact, that quantization of the interacting relativistic field theories leads to infinities — appearence of the divergent integrals which are dealt with by the proccess of renormalization. Traditionally these infinities were considered as a plague of the Quantum Field Theory. One can find very strong words denouncing them, belonging to the great figures of several generations, such like Dirac, Feynman and others. However I shall try to show, that the infinities in the Yang-Mills theory are beneficial — they lead to appearence of the dimensional parameter after the quantization of this theory.

This point of view was already emphasized by R. Jackiw [8] but to my knowledge it is not shared yet by other specialists.

Sidney Coleman [9] coined a nice name "dimensional transmutation" for the phenomenon, which I am going to describe. Let us see what all this means.

3 Dimensional transmutation

The most direct way to see, how "infinities" appear in quantum Yang-Mills theory, is to begin evaluation of the functional integral (1) in some approximate fashion. The most evident is the “stationary phase” method. We put

$$A_\mu = A_\mu^{\mathrm{cl}} + a_\mu ,$$

where $A^{\rm cl}_\mu$ is a solution of classical equation of motion

$$\nabla_\mu F_{\mu\nu} = \partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] = 0$$

with prescribed asymptotic conditions and leave quadratic form in a in the action. Integral then is of Gaussian type and reduces to determinants.

In fact to take into account the integral over gauge orbit in this situation, it is more appropriate to change the "gauge fixing" terms in the action, namely substitute $\int \mathrm{tr}(\partial_\mu A_\mu)^2 dx$ by $\int \mathrm{tr}(\nabla^{\rm cl}_\mu a_\mu)^2 dx$, where

$$\nabla^{\rm cl}_\mu \cdot = \partial_\mu \cdot + [A^{\rm cl}_\mu, \cdot\,].$$

Corresponding normalizing determinant will induce the term

$$\int \mathrm{tr}(\nabla^{\rm cl}_\mu \bar{c}_\mu \nabla_\mu c)^2 d^4x$$

into action. Indeed, with this convention terms, linear in a will not appear.

The gaussian integral obtained in this approximation looks as follows

$$Z(A_{\rm as}) = \exp\{i\mathcal{A}(A^{\rm cl})\} \int \exp\{i[(M_1 a, a) + (\bar{c}, M_0 c)]\} \prod da\, d\bar{c}\, dc\,,$$

where M_1 and M_0 are second order differential operators

$$(M_1 a)_\nu = (\nabla^{\rm cl}_\mu)^2 a_\nu + 2[F^{\rm cl}_{\mu\nu}, a_\mu];$$
$$M_0 c = (\nabla^{\rm cl}_\mu)^2 c.$$

The logarithm of the functional Z is called the "effective action" and denoted by $W(A)$. We have

$$W(A^{\rm cl}) = \frac{1}{i} \ln Z(A^{\rm cl}) = \mathcal{A}(A^{\rm cl}) - \frac{1}{2} \ln\det M_1 + \ln\det M_0.$$

It is clear from the definition, that the functional $W(A^{\rm cl})$ is manifestly gauge invariant with respect to gauge transformation of $A^{\rm cl}$.

There are many ways to evaluate the determinants of the differential operators. We shall not discuss them here in detail, refering to physics text-books, e. g. [7].

However several highlights deserve to be mentioned. First of all we can represent M_1 and M_0 as a perturbation of the laplacian, e. g.

$$M_1 = \Box + K_\mu \partial_\mu + L,$$

where $\Box$ is Laplacian

$$\Box = \partial_\mu^2 \cdot$$

and K_μ and L — matrices, acting on a_μ^a, expressed via A_μ^{cl} and its first derivatives. Using evident formula

$$\ln\det M_1 = \ln\det\Box + \ln\det(I + \Box^{-1}(K\partial + L))$$

we can drop the first term in RHS as an irrelevant (though divergent) constant and thus essentially regularize the determinant. However the second term in the RHS is still divergent due to the singular nature of the Green function $\Box^{-1}$. Using convenient formula

$$\begin{aligned}\ln\det(I + \Box^{-1}(K\partial + L)) &= \mathrm{Tr}\ln(I + \Box^{-1}(K\partial + L)) \\ &= \sum \frac{(-1)^n}{n}\,\mathrm{Tr}\left(\Box^{-1}(K\partial + L)\right)^n \qquad (2)\end{aligned}$$

we see, that several first terms in this expansion contain the divergent integrals. We use notation Tr for the functional trace to distinguish it from tr in the Lie algebra.

For example the term of second order in the expression (2) contains the expression

$$\mathrm{Tr}(\Box^{-1}L\,\Box^{-1}L) = \iint D^2(x-y)\,\mathrm{tr}\,L(x)L(y)\,d^4x\,d^4y\,, \qquad (3)$$

where $D(x-y)$ is a Green function of Laplacian. In four dimensional space $D(x)$ has singularity in the vicinity of origine

$$D(x) \sim \frac{1}{(x,x)}$$

so that integral (3) diverges logarithmically. There are terms in the expansion which look to be divergent even more severely, but a careful treatment show, that

1) Only several lower order terms in expansion (2) are divergent.

2) Only logarithmic divergences are present.

3) The divergent terms depend on A^{cl} only in local way, so that these terms are proportional to $\int P(x)dx$ where $P(x)$ is polinomial in A and its derivatives.

Let us illustrate the last point on the example of the integral (3). Rewriting

$$\operatorname{tr} L(x)L(y) = \operatorname{tr}(L(y) - L(x))L(x) + \operatorname{tr} L^2(x)$$

we see, that the first term in the RHS leads to convergent integral and the second one gives

$$\int \operatorname{tr} L^2(x) d^4(x) \cdot \int \frac{d^4 y}{(y, y)^2}$$

More careful treatment shows that only the divergence of the last integral at $y = 0$ is relevant. Introducing cutoff $(y)^2 > \varepsilon^2$ we get finally the expression

$$\int \operatorname{tr} L^2(x)\, d^4(x) \cdot \ln \frac{1}{\varepsilon m}$$

as a divergent part, where the divergent log is multiplied by a local term. Note, that we were to introduce another parameter m of dimension $[\mathrm{L}]^{-1}$ to be able to write logarithm. This extra parameter characterize the regularization of the integral. We shall see soon, that it has fundamental importance.

Usually small space cutoff ε is substituted by large momentum cutoff $\Lambda = 1/\varepsilon$, which would appear if we decided to calculate the terms in (2) via Fourier transform. We shall use this convention in what follows.

Now let us invoke the gauge invariance of $W(A^{\mathrm{cl}})$. The only local dimensionless and gauge invariant functional of A is classical action. This means, that $W(A^{\mathrm{cl}})$ gets the form

$$W(A^{\mathrm{cl}}) = \left(\frac{1}{g^2} + c \ln \frac{\Lambda}{m}\right) \mathcal{A}_{\mathrm{cl}} + \text{finite terms}. \tag{4}$$

Parameter g^2 does not enter finite terms, however they essentially depend on the "normalization" parameter m. Now the most important property of the Yang-Mills theory is that the constant c in (4) is negative. For the case of $G = SU(2)$

$$c = -\frac{1}{8\pi^2}\,\frac{11}{3}\,.$$

Famous calculation of this result was done in the beginning of 70-ties [10] and led to ressurection of Quantum Field Theory in the minds of physicists. The reason for this can be found in the textbooks I already refered to. For our goal it is important in the following sence. We see, that we can define a finite expression for the effective action if we allow the coupling constant g^2 depend on Λ in such a way, that

$$\frac{1}{g^2(\Lambda)} + c \ln \frac{\Lambda}{m} = 0 \tag{5}$$

and go to the limit $\Lambda \to \infty$. The negativeness of c is crucial for such a limit to make sence.

The finite terms then define the physical effective action. It does not depend on the original dimensionless coupling constant g^2. Instead, it depends on a new parameter m having dimension of mass. It is the new effective action which should have physical interpretation, and now it has chance to lead to massive particle spectrum as it contains the dimensional parameter. Of course I did not show in any way how to describe this spectrum. Real work (which should lead to the solution of Clay Problem) must be based on the control of full effective action, for which our expressions give only "one loop" approximation. However, I hope, that I was able to indicate the important property of Yang-Mills theory: its quantum version is very different from the classical one. Regularization of the theory may be done, but the conformal invariance of classical theory is brocken. A dimensionless coupling constant of classical action is traded for the dimensional parameter in quantum effective action. Moreover, through this process of regularization and "dimensional transmutation" the effective action can be introduced without divergences, defining the correct quantum Yang-Mills theory.

4 A simple mathematically clear example of "dimensional transmutation"

I shall give an explicite example in which initially divergent (and thus seemingly meaningless) problem can be regularized in such a way, that dimensional transmutation takes place and the limiting theory has well understood meaning.

The example employs the Schröedinger operator

$$H = H_0 + V; \quad H_0\psi = -\triangle\psi, \quad V\psi = v(x)\psi,$$

acting on function $\psi(x)$ on the plane $\mathbb{R}^2$. The potential $v(x)$ is taken to be "point-like"

$$V(x) = \varepsilon\delta^{(2)}(x),$$

where $\delta^{(2)}(x)$ is a δ-function. It is clear that both terms in H have the same dimension $[\mathrm{L}]^{-2}$, do that the "coupling constant" ε is dimensionless.

In treating the formal expression for H we encounter the infinity. Let us exploit one way to see this and construct the resolvent of H

$$R(z) = (H - zI)^{-1}.$$

The standard formulas of scattering theory (see e. g. [11]) tell us, that $R(z)$ has structure

$$R(z) = R_0(z) - R_0(z)T(z)R_0(z),$$

where $T(z)$ satisfies the equation

$$T(z) = V - VR_0(z)T(z).$$

Let us write this equation in the momentum representation (use Fourier transform). In this representation the kernel of operator V is a constant

$$v(p, p') = \varepsilon$$

and kernel $t(p, p'; z)$ of operator $T(z)$ does not depend on p, p' either

$$t(p, p'; z) = t(z).$$

The equation for $T(z)$ takes the form

$$t(z) = \varepsilon - \varepsilon \int \frac{d^2p}{p^2 - z} t(z)$$

or

$$\frac{1}{t(z)} = \frac{1}{\varepsilon} + \int \frac{d^2p}{p^2 - z}.$$

The integral in the RHS diverges at large $|p|$, thus the "infinity" appears in the construction of the resolvent $R(z)$. Introducing cutoff $|p| < \Lambda$ we get

$$\frac{1}{t(z)} = \frac{1}{\varepsilon} + \pi \ln \frac{\Lambda^2}{-z} = \frac{1}{\varepsilon} + \pi \ln \frac{\Lambda^2}{m^2} + \pi \ln \frac{m^2}{-z}.$$

Now if we take ε to be negative (case of attraction) we can go to the limit $\Lambda \to \infty$, $\varepsilon \to -0$ in such a way that

$$\frac{1}{\varepsilon} + \pi \ln \frac{\Lambda^2}{m^2} = 0, \qquad (6)$$

so that we get for the limiting theory

$$t(z) = \frac{1}{\pi} \cdot \frac{1}{\ln m^2 / - z}.$$

We see, that the formula (6) is exactly the same as (5), so that the dimensional transmutation in Yang-Mills theory and this example of point-like interaction is the same. This was already observed by R. Jackiw [8], as was mentioned above.

Now, in this case the new dimensional parameter has a simple interpretation. The function $t(z)$ produces the simple pole for resolvent $R(z)$ and $z = -m^2$ corresponds to discreet spectrum. Thus physically the interaction produces a bound state and dimensonal parameter m^2 is its energy.

It is important to know, that the constructed operator $R(z)$ is indeed a resolvent of some self-adjoint operator H_{reg}. Indeed, we have involution

$$R^*(z) = R(\bar{z})$$

and it is easy to check the Hilbert identity

$$R(z_1) - R(z_2) = (z_1 - z_2) R(z_1) R(z_2).$$

The mathematically clear interpretation of this operator H_{reg} was given by Berezin and me [12] long ago. If one considers operator $\hat{H}_0$, defined by the Laplacian on the domain, consisting of functions $\psi(x)$ vanishing in the vicinity of $x = 0$, it will not be essentially selfadjoint. Rather it will have indices (1,1) allowing for a one parameter family of selfadjoint extensions. The operator $R(z)$ is exactly a resolvent of such an extension and m is a corresponding parameter.

This consideration gives a mathematical validation to the process of regularization and consequent dimensional transmutation. It presumably should help to persuade the reader to believe, that our manipulations with Yang-Mills theory eventually are to get a mathematical sence also.

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